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Examples of presentations which are minimally Cockroft in several different ways

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For Karl Gruenberg on his 65th birthday

1. Introduction

Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ be a group presentation, and let G be the group defined by $\mathcal{P}$. If we regard $\mathcal{P}$ as a 2-complex with one 0-cell, a 1-cell for each $x \in \mathbf{x}$, and a 2-cell for each $R \in \mathbf{r}$ in the standard way, then G is just the fundamental group $\pi_1(\mathcal{P})$ of $\mathcal{P}$. There is also, of course, the second homotopy group $\pi_2(\mathcal{P})$, which is a left $\mathbb{Z}G$-module. The elements of $\pi_2(\mathcal{P})$ can be represented by geometric configurations called *spherical pictures*, as described in [5]. (It will be convenient in this paper to allow only one basepoint on each disc in our pictures, so our pictures will actually be what are called $*$-pictures in [5].)

There is a standard embedding μ of $\pi_2(\mathcal{P})$ into the free left $\mathbb{Z}G$-module $\bigoplus_{R \in \mathbf{r}} \mathbb{Z}Ge_R$ defined as follows. (For further details see [5].) Consider an element ξ of $\pi_2(\mathcal{P})$ represented by a spherical picture $\mathbb{P}$, where $\mathbb{P}$ has discs $\Delta_1, \ldots, \Delta_n$ with labels $R_1^{\varepsilon_1}, \ldots, R_n^{\varepsilon_n}$ ($R_i \in \mathbf{r}$, $\varepsilon_i = \pm 1$ for $i = 1, \ldots, n$). Choose a point 0 outside $\mathbb{P}$, and let γ_i be a transverse path from 0 to the basepoint of Δ_i. The label on γ_i represents an element g_i of G ($i = 1, \ldots, n$), and we define $\mu(\xi)$ to be

$$\sum_{i=1}^{n} \varepsilon_i g_i e_{R_i}.$$

If H is a subgroup of G, then we will say that $\mathcal{P}$ has the *left* (resp. *right*)

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H-identity property if for each $\xi \in \pi_2(\mathcal{P})$ all the coefficients of $\mu(\xi)$ lie in $\mathbb{Z}G.IH$ (resp. $IH.\mathbb{Z}G$).

When H is normal in G then the left and right H-identity properties coincide, and we can refer simply to the *H-identity property*.

A presentation which has the 1-identity property is said to be *aspherical* (this just means that $\pi_2(\mathcal{P}) = 0$). A presentation with the G-identity property is said to be *Cockroft*. More generally, a presentation with the *right* H-identity is said to be *H-Cockroft*. This property arises in connection with the Whitehead conjecture, and has received considerable attention (see [2] and the references cited there). On the other hand, the *left* H-identity property arises quite naturally if one thinks about formulating a generalization of the 1-identity property in terms of pairings of terms of identity sequences (or equivalently, pairings of discs of spherical pictures). (See Section 2.)

Let $\mathcal{H}^{(l)}$ (resp. $\mathcal{H}^{(r)}$) be the set of subgroups H of G such that $\mathcal{P}$ has the left (resp. right) H-identity property. It is clear that if $H \in \mathcal{H}^{(l)}$ and if $K \supseteq H$ then $K \in \mathcal{H}^{(l)}$, and similarly for $\mathcal{H}^{(r)}$. Harlander [4] has shown that the poset $\mathcal{H}^{(r)}$ has minimal elements. An alternative proof has been given by Gilbert and Howie [3]. Their proof also serves to show that $\mathcal{H}^{(l)}$ has minimal elements. (In the case when $\pi_2(\mathcal{P})$ is finitely generated as a module, this in fact is clear from the discussion in Section 2, since then the set $\mathcal{H}_0^{(l)}$ considered there is finite.)

It has been an open question whether there are presentations for which $\mathcal{H}^{(r)}$ has more than one minimal element. Examples will be given here to show that this can indeed be the case. We will also consider the minimal elements of $\mathcal{H}^{(l)}$ for these examples. In fact, we will begin with a discussion of $\mathcal{H}^{(l)}$.

2. The left identity property

Suppose that $\mathcal{P}$ is Cockroft in the standard sense (i.e. G-Cockroft). This means that any spherical picture $\mathbb{P}$ over $\mathcal{P}$ has an even number of discs, say $2m$, and that there is a pairing

$$\vartheta : \Delta_i \leftrightarrow \Delta_i' \quad (i = 1, \ldots, m)$$

of the discs of $\mathbb{P}$ such that Δ_i and Δ_i' have labels which are inverse to each other. For such a pairing, draw a transverse path from the basepoint of Δ_i to the basepoint of Δ_i', and let h_i be the element of G represented by the label on this path. Let

$$H_\vartheta = \mathrm{sgp}_G\{h_i : i = 1, \ldots, m\} .$$

Now let $X = \{\mathbb{P}_j : j \in J\}$ be a collection of spherical pictures which represent a set of module generators of $\pi_2(\mathcal{P})$, and for each $j \in J$ choose a pairing of ϑ_j of $\mathbb{P}_j$, giving a pairing $\vartheta = \{\vartheta_j\}_{j \in J}$ for the whole collection X. Let

$$L_\vartheta = \mathrm{sgp}_G\{H_{\vartheta_j}\colon j \in J\}\,.$$

Then it is clear that $\mathcal{P}$ has the left L_ϑ-identity property, so that the set

$$\mathcal{H}_0^{(l)} = \{L_\vartheta\colon \vartheta \text{ a pairing for } X\}$$

is contained in $\mathcal{H}^{(l)}$.

Moreover, suppose $\mathcal{P}$ has the left H-identity property. Then for *any* spherical picture over $\mathcal{P}$ there must be a pairing ρ of the discs so that the subgroup H_ρ of G is contained in H. In particular, this must happen for each $\mathbb{P}_j$, so there must be a pairing ϑ for the collection X such that $L_\vartheta \subseteq H$.

Suppose therefore that we can show that, if ϑ, ϑ' are distinct pairings for X, then L_ϑ and $L_{\vartheta'}$ are incomparable, i.e.

$$L_\vartheta \not\subseteq L_{\vartheta'}\,, \qquad L_{\vartheta'} \not\subseteq L_\vartheta\,. \tag{1}$$

Then the minimal elements of $\mathcal{H}^{(l)}$ will be precisely the elements of $\mathcal{H}_0^{(l)}$.

Note that in order to show that (1) holds, it suffices to find a homomorphic image $\bar{G}$ of G such that the images of L_ϑ and $L_{\vartheta'}$ are incomparable.

3. An example

Let $\mathcal{P} = \langle a, b, t;\, R, S, T\rangle$ where

$$R = aba^{-1}b^{-1}\,, \quad S = a^2ta^{-2}t^{-1}\,, \quad T = b^2tb^{-2}t^{-1}\,.$$

It follows from [1] that $\pi_2(\mathcal{P})$ is generated by a single element represented by the spherical picture $\mathbb{P}_0$ in Fig. 1.

There are two S-discs (which have a unique pairing), two T-discs (unique pairing), and eight R-discs which can be paired in 4! ways. This gives 24 groups in $\mathcal{H}_0^{(l)}$.

Now let $\bar{G}$ be the image of G obtained by setting a^2 and b^2 equal to 1. Then $\bar{G} = V * \langle t\rangle$, where V is Klein's 4-group (generated by a and b). The images of the elements of $\mathcal{H}_0^{(l)}$ in $\bar{G}$ are the groups

$$\bar{H}_\sigma = \mathrm{sgp}_{\bar{G}}\{1t\sigma(1),\, at\sigma(a),\, bt\sigma(b),\, abt\sigma(ab)\}\,,$$

where σ is a permutation of the four elements 1, a, b, ab of V. Clearly $\bar{H}_\sigma$ is free on the given generators. In particular, the only elements of t-length 1 in $\bar{H}_\sigma$ are

$\mathbb{P}_0$:

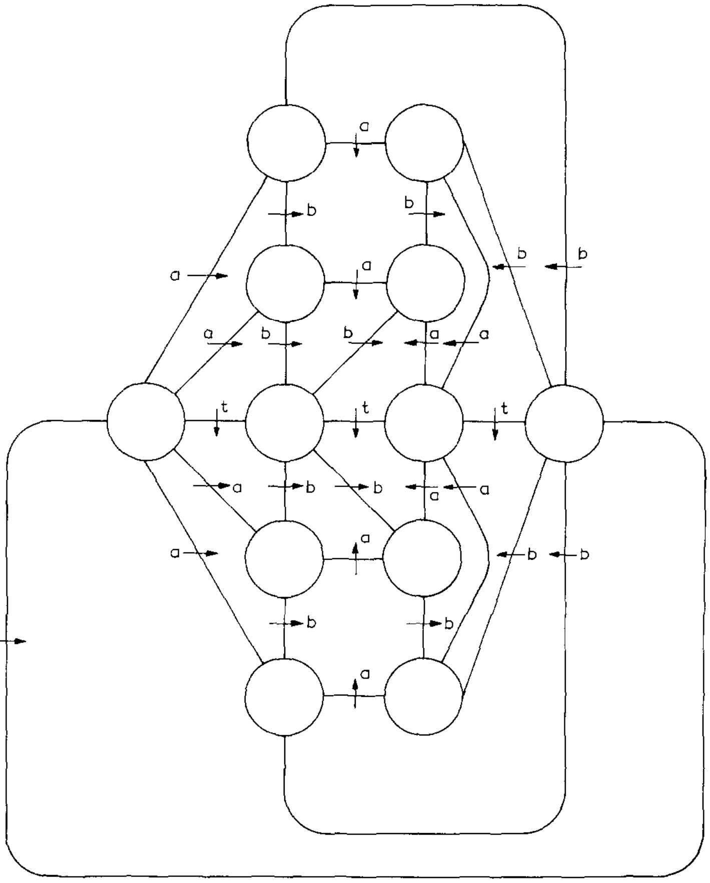

Fig. 1.

the generators (and their inverses). It follows that if $\sigma \neq \tau$ then $\bar{H}_\sigma$ and $\bar{H}_\tau$ are incomparable.

Remark. The above can be generalized. We take

$$\mathscr{P}_k = \langle a, b, t;\, R, S_k, T_k \rangle \tag{2}$$

with

$$R = aba^{-1}b^{-1}\,, \quad S_k = a^k t a^{-k} t^{-1}\,, \quad T_k = b^k t b^{-k} t^{-1}$$

for $k \geq 1$. Then $\pi_2(\mathscr{P})$ is generated by a single element represented by a spherical picture with two S_k-discs, two T_k-discs, and $2k^2$ R-discs. There are $(k^2)!$ pairing for this picture giving rise to $(k^2)!$ distinct minimal elements of $\mathscr{H}^{(l)}$.

4. The right identity property

The difference between looking at the left identity property and the right identity property is that for the latter it is not enough to restrict attention to module generators of $\pi_2(\mathscr{P})$, since $\mathbb{Z}H.IG$ is not in general a left $\mathbb{Z}G$-module. We must instead look at all translates of module generators by elements of G.

Rather than going in to generalities, we concentrate on our example, so let $\mathscr{P}$ and $\mathbb{P}_0$ be as in Section 3. For any function φ from G to $\{1, a, b, ab\}$ let

$$K_\varphi = \mathrm{sgp}_G\{a^2, b^2, gt\varphi(g)^{-1}g^{-1} \ (g \in G)\} .$$

Now the image of the module generator of $\pi_2(\mathscr{P})$ under the standard embedding into $\mathbb{Z}Ge_R \oplus \mathbb{Z}Ge_S \oplus \mathbb{Z}Ge_T$ is

$$(1 - b^2)e_S + (a^2 - 1)e_T + (t - 1)(1 + a + b + ab)e_R .$$

Considering all the translates of this by elements of G, we see that if H is a subgroup of G then $\pi_2(\mathscr{P})$ has the right H-identity property if and only if

$$g(a^2 - 1) \in IH.\mathbb{Z}G , \tag{3}$$

$$g(b^2 - 1) \in IH.\mathbb{Z}G , \tag{4}$$

$$g(t - 1)(1 + a + b + ab) \in IH.\mathbb{Z}G , \tag{5}$$

for all $g \in G$.

Lemma. *In order for* (3), (4) *and* (5) *to hold it is necessary and sufficient that* $K_\varphi \subseteq H$ *for some* φ.

Proof. Necessity is clear.

For sufficiency, note that since a^2 and b^2 are central in G, if $a^2, b^2 \in H$ then (3) and (4) hold. Now consider (5). By assumption, $Hgt = Hg\varphi(g)$ $(g \in G)$. Thus

$$Hgta = Hg\varphi(g)a , \quad Hgtb = Hg\varphi(g)b , \quad Hgtab = Hg\varphi(a)ab .$$

Now, since a^2, b^2 are assumed to belong to H (and are central), $(Hg\varphi(g)$, $Hg\varphi(g)a$, $Hg\varphi(g)b$, $Hg\varphi(g)ab)$ is just a permutation of $(Hg, Hga, Hgb, Hgab)$, so (5) holds. □

Corollary. *If no conjugate of* a, b *or* ab *belongs to* K_φ, *then* K_φ *does not contain any* K_ψ *for* $\psi \neq \varphi$.

Proof. Suppose $K_\psi \subseteq K_\varphi$, with $\psi \neq \varphi$. Let g be such that $\psi(g) \neq \varphi(g)$. Then

$$g\psi(g)\varphi(g)^{-1}g^{-1} \in K_\varphi .$$

But $g\psi(g)\varphi(g)^{-1}g^{-1}$ is equal modulo $\mathrm{sgp}_G\{a^2, b^2\}$ (and hence equal modulo K_φ) to one of gag^{-1}, gbg^{-1}, $gabg^{-1}$.

Corollary. *The subgroups*

$$K_\varphi \quad (\varphi \text{ a mapping from } G \text{ to } \{1, a, b, ab\}, \\ K_\varphi \text{ contains no conjugate of } a, b \text{ or } ab)$$

are distinct minimal elements of $\mathcal{H}^{(r)}$. □

In particular, by considering the four constant functions from G to $\{1, a, b, ab\}$ we obtain four distinct minimal elements of $\mathcal{H}^{(r)}$.

Remark. If we consider the presentation $\mathcal{P}_k$ in (2) we obtain an example for which $\mathcal{H}^{(r)}$ has at least k^2 distinct minimal elements.

Acknowledgment

In my original work on this subject I identified the H-Cockroft property with what I have called the *left* H-identity property, rather than the right H-identity property. (This confusion between left and right had been present in some recent literature on the subject.) I am extremely grateful to J. Howie for pointing out this left/right distinction, and for indicating that my arguments for the left H-identity property could be modified to deal with the right H-identity property. I am also grateful to M. Dyer and N. Gilbert for their contributions to this left/right discussion.

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